

Conformal Invariance Of Interacting WZNW Models

Oleg A. Soloviev ^{*†}

*Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, United Kingdom*

Abstract

We consider two level k WZNW models coupled to each other through a generalized Thirring-like current-current interaction. It is shown that in the large k limit, this interacting system can be presented as a two-parameter perturbation around a nonunitary WZNW model. The perturbation operators are the sigma model kinetic terms with metric related to the Thirring coupling constants. The renormalizability of the perturbed model leads to an algebraic equation for couplings. This equation coincides with the master Virasoro equation. We find that the beta functions of the two-parameter perturbation have nontrivial zeros depending on the Thirring coupling constants. Thus we exhibit that solutions to the master equation provide nontrivial conformal points to the system of two interacting WZNW models.

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^{*}e-mail: soloviev@V1.PH.QMW.ac.uk

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1 Introduction

At present, the importance of two dimensional conformal field theories (CFT's) grows rapidly. It is being comprehended that the methods of CFT's provide powerful tools to approach many physical problems from black holes [1],[2] to turbulence [3],[4]. Besides, CFT's continue to be a central element of string theory.

Among all CFT's, those which admit a proper Lagrangian formulation appear to be of an ultimate significance both in string theory and statistical physics. For example, the minimal conformal models describing many statistical systems [5],[6], possess a very elegant Lagrangian description in terms of gauged WZNW models [7],[8]. Also, in string theory, many exact string backgrounds are obtained from gauged WZNW models [1],[9]-[24]. These examples indicate the importance of Lagrangian CFT's formulated in terms of gauged WZNW models.

The interesting observation is that gauged WZNW models in turn can be properly understood as ordinary WZNW theories [25]-[27] coupled to each other through the (isoscalar) Thirring-like current-current interaction at a particular value of the Thirring coupling constant [28]. However, the isoscalar coupling is the simplest type of general renormalizable current-current interactions. Hence, it is natural to assume that at different values of Thirring coupling constants, the more general Thirring current-current interaction ought to give rise to new CFT's. It is noteworthy that WZNW models with arbitrary current-current interaction emerge naturally within the coadjoint orbit method [29] as well as in the theory of chiral WZNW models [30]-[32]. This makes the above conjecture plausible.

The aim of the present paper is to shed some light on the issue of conformal invariance of WZNW models with current-current interactions. We will consider two unitary level k WZNW models coupled to each other through a generic Thirring-like current-current interaction. We will exhibit that there are special values of the Thirring coupling constants at which the interacting systems can be understood as infrared conformal points of a certain nonconformal theory. These critical values of the couplings will be shown to be connected to solutions of the master Virasoro equation [33],[34]. Thus we will give

more evidence for the existence of an intimate relation between Thirring models and the affine-Virasoro construction [35].

The nonconformal theory whose infrared critical points we are going to study in this paper is the nonunitary WZNW model perturbed by two orthogonal quasimarginal relevant operators. These perturbations are generalizations of the conformal operators discussed in [36]-[38]. By establishing the link between infrared conformal points of the perturbed nonunitary WZNW theory and interacting unitary WZNW models, we will prove the conformal symmetry of the latter.

The paper is organized as follows. In section 2 the definition of interacting WZNW models is given. In section 3 the action of two interacting unitary level k WZNW models is presented in a factorized form which will be convenient for implementing the $1/k$ method. In section 4 we will consider a two-parameter perturbation around the nonunitary WZNW model. We will exhibit that at its infrared conformal points the perturbed theory coincides with the system of two interacting WZNW models with the coupling constants defined by solutions of the master-Virasoro equation. Finally, in the last section we conclude with some comments on the results obtained.

2 Interacting WZNW models

Let $S_{WZNW}(g_1, k)$ and $S_{WZNW}(g_2, k)$ be the actions of two conformal level k WZNW models. Let us consider the following interaction between these two WZNW models*.

$$S(g_1, g_2, S) = S_{WZNW}(g_1, k) + S_{WZNW}(g_2, k) + S_I(g_1, g_2, S), \quad (2.1)$$

where these three terms respectively are given by

$$\begin{aligned} S_{WZNW}(g_1, k) &= \frac{-k}{4\pi} \left\{ \int \text{Tr} |g_1^{-1} dg_1|^2 + \frac{i}{3} \int d^{-1} \text{Tr} (g_1^{-1} dg_1)^3 \right\}, \\ S_{WZNW}(g_2, k) &= \frac{-k}{4\pi} \left\{ \int \text{Tr} |g_2^{-1} dg_2|^2 + \frac{i}{3} \int d^{-1} \text{Tr} (g_2^{-1} dg_2)^3 \right\}, \\ S_I &= \frac{-k^2}{\pi} \int d^2 z \text{Tr}^2 (g_1^{-1} \partial g_1 S \bar{\partial} g_2 g_2^{-1}), \end{aligned} \quad (2.2)$$

*A generalization to three and more interacting WZNW models is straightforward.

with the coupling S belonging to the direct product of two Lie algebras $\mathcal{G}_1 \otimes \mathcal{G}_2$. Here the fields g_1 and g_2 take their values in the Lie groups G_1 and G_2 respectively. In what follows we will suppose $G_1 = G_2 = G$. The symbol Tr^2 indicates a double tracing over the indices of a matrix from the tensor product $\mathcal{G}_1 \otimes \mathcal{G}_2$.

By construction, the classical theory in eq. (2.1) is invariant under conformal transformations $z \rightarrow f(z)$, $\bar{z} \rightarrow \bar{f}(\bar{z})$. Indeed, the coupling matrix S is dimensionless. Therefore, the classical interaction term is conformally invariant by itself. In addition, there are affine symmetries defined by the following transformations [39]

$$g_1 \rightarrow \bar{\Omega}_1(\bar{z})g_1h_1\Omega_1(z)h_1^{-1}, \quad g_2 \rightarrow h_2^{-1}\bar{\Omega}_2(\bar{z})h_2g_2\Omega_2(z), \quad (2.3)$$

where the functions h_1 , h_2 are computed from the equations

$$\bar{\partial}h_1h_1^{-1} = 2k\text{Tr } S \bar{\partial}g_2g_2^{-1}, \quad h_2^{-1}\partial h_2 = 2k\text{Tr } S g_1^{-1}\partial g_1. \quad (2.4)$$

Parameters $\Omega_{1,2}$ and $\bar{\Omega}_{1,2}$ are arbitrary functions of z and \bar{z} respectively.

The equations of motion of the theory described by the action in eq. (2.1) coincide with the analyticity conditions of the local affine currents

$$\partial\bar{\mathcal{J}}_1 = 0, \quad \bar{\partial}\mathcal{J}_2 = 0, \quad (2.5)$$

where

$$\bar{\mathcal{J}}_1^{\bar{a}} = \bar{J}_1^{\bar{a}} + k\phi_1^{a\bar{a}} S^{a\bar{b}} \bar{J}_2^{\bar{b}}, \quad (2.6)$$

$$\mathcal{J}_2^a = J_2^a + k\phi_2^{a\bar{a}} S^{b\bar{a}} J_1^b.$$

Here we have used the following notations

$$\begin{aligned} J_1^a &= -\frac{k}{2}\text{Tr}(g_1^{-1}\partial g_1 t^a), \\ J_2^a &= -\frac{k}{2}\text{Tr}(g_2^{-1}\partial g_2 t^a), \\ \bar{J}_1^{\bar{a}} &= -\frac{k}{2}\text{Tr}(\bar{\partial}g_1g_1^{-1}t^{\bar{a}}), \\ \bar{J}_2^{\bar{a}} &= -\frac{k}{2}\text{Tr}(\bar{\partial}g_2g_2^{-1}t^{\bar{a}}), \\ \phi_1^{a\bar{a}} &= \text{Tr}(g_1 t^a g_1^{-1} t^{\bar{a}}), \\ \phi_2^{a\bar{a}} &= \text{Tr}(g_2^{-1} t^a g_2 t^{\bar{a}}), \end{aligned} \quad (2.7)$$

where t^a are the generators of the Lie algebra \mathcal{G} associated with the Lie group G ,

$$[t^a, t^b] = f^{abc}t^c, \quad (2.8)$$

with f^{abc} the structure constants.

In general, the coupling matrix S in eq. (2.1) can be chosen arbitrarily. However, in what follows we will put restrictions on S to be symmetrical and invertible. In this case, the bosonic theory described by the action (2.1) is equivalent to the non-Abelian fermionic Thirring model [35]. Also, it is interesting to point out that the equations of motion (2.6) allow us to express any derivative of $J_{1,2}$, $\bar{J}_{1,2}$ in terms of the fields $\phi_{1,2}$ and $J_{1,2}$, $\bar{J}_{1,2}$ without derivatives.

As we have mentioned above, the interaction between two WZNW models in eq. (2.1) with arbitrary S preserves the conformal invariance at the classical level. However, in the course of quantization of the interacting theory, the conformal symmetry can be broken. This can be seen on a simple example $G = SU(2)$, $k = 1$ and $S = \lambda I$, where I is unity in $su(2) \otimes su(2)$. When λ is small, the theory given by (2.1) is equivalent to the sine-Gordon model [40] which is not a conformal theory. At the same time, there may exist some special values of the coupling matrix S at which the conformal invariance extends to the quantum level. The following example illustrates such a possibility. Let us take $S = I/2k$. With the given S one can use the Polyakov-Wiegmann formula [28] to obtain

$$S(g_1, g_2, S = \frac{I}{2k}) = S_{WZNW}(f, k), \quad (2.9)$$

where $f = g_1 g_2$. The conformal invariance of this theory follows immediately from the fact that the right hand side of eq. (2.9) is a conformal WZNW model. Of course, the Polyakov-Wiegmann conformal point is very peculiar because it has the extra gauge invariance under $g_1 \rightarrow g_1 \Lambda$, $g_2 \rightarrow \Lambda^{-1} g_2$, where Λ is arbitrary nondegenerate matrix. Nevertheless, the considered example signifies that nontrivial conformal points in the theory (2.1) may actually exist. The rest of the paper will be dedicated to clarifying this conjecture.

3 Expansion in the couplings

Let us assume that the coupling matrix S can be presented in the following form

$$S = \sigma \hat{S}, \quad (3.10)$$

where σ is a small parameter. Apparently, the interaction term in eq. (2.1) is linear in σ . We will show that this theory can be, in fact, recast in a new form without interaction between g_1 and g_2 but instead with a highly nonlinear dependence on σ .

Let us make in the theory (2.1) the following change of variables

$$\begin{aligned} g_1 &\rightarrow \tilde{g}_1, \\ g_2 &\rightarrow h(\tilde{g}_1) \cdot \tilde{g}_2, \end{aligned} \quad (3.11)$$

where the function $h(\tilde{g}_1)$ is the solution of the following equation

$$\partial h \cdot h^{-1} = -2k\sigma \text{Tr} \hat{S} \tilde{g}_1^{-1} \partial \tilde{g}_1. \quad (3.12)$$

Since h is a nonlocal function of \tilde{g}_1 , the functional of the new variable \tilde{g}_1 is going to be nonlocal as well.

One can check that the action given by eq. (2.1) in the new variables \tilde{g}_1, \tilde{g}_2 takes the form

$$\begin{aligned} S(g_1, g_2, k) &\rightarrow S(\tilde{g}_1, \tilde{g}_2, k) = S_{WZ\text{NW}}(\tilde{g}_2, k) + S_{WZ\text{NW}}(\tilde{g}_1, k) \\ &+ S_{WZ\text{NW}}(h, k) - \frac{k^2\sigma}{\pi} \int d^2z \text{Tr}^2 \tilde{g}_1^{-1} \partial \tilde{g}_1 \hat{S} \bar{\partial} h h^{-1}. \end{aligned} \quad (3.13)$$

Thus, after the change, the new field \tilde{g}_2 completely decouples from \tilde{g}_1 .[†] The price we pay for this factorization is a highly nonlocal theory of the variable \tilde{g}_1 . The last theory is described by the following nonlocal functional

$$S(\tilde{g}_1) = S_{WZ\text{NW}}(\tilde{g}_1, k) + S_{WZ\text{NW}}(h(\tilde{g}_1), k) - \frac{k^2\sigma}{\pi} \int d^2z \text{Tr}^2 \tilde{g}_1^{-1} \partial \tilde{g}_1 \hat{S} \bar{\partial} h(\tilde{g}_1) h^{-1}(\tilde{g}_1). \quad (3.14)$$

[†]The important point to be made is that the Jacobian of the change of variables in eq. (3.11) is equal to one. Therefore, the factorized action given by eq. (3.13) holds at the quantum level as well.

The given model is very complicated, because of the function h . The latter can be found by iterations from eq. (3.12). Moreover, the final result can be expressed in terms of the functions \tilde{J}_1 , $\bar{\tilde{J}}_1$ and $\tilde{\phi}_1$. There is a useful formula

$$\bar{\partial} h h^{-1}(z, \bar{z}) = -2k\sigma \int d^2 y \bar{\partial}_{\bar{z}} G(z, y) \{ \text{Tr } \hat{S} \tilde{g}_1^{-1}(y, \bar{y}) \partial_y \left(\bar{\partial}_{\bar{y}} \tilde{g}_1(y, \bar{y}) \tilde{g}_1^{-1}(y, \bar{y}) \right) \tilde{g}_1(y, \bar{y}) \quad (3.15)$$

$$+ [\text{Tr } \hat{S} \tilde{g}_1^{-1}(y, \bar{y}) \partial_y \tilde{g}_1(y, \bar{y}), \bar{\partial} h(y, \bar{y}) h^{-1}(y, \bar{y})] \}.$$

From this equation, one can express $\bar{\partial} h h^{-1}$ in terms of \tilde{g}_1 . The Green function $G(z, y)$ in eq. (3.15) satisfies the following relation

$$\bar{\partial} \partial G(z, y) = \delta^{(2)}(z, y). \quad (3.16)$$

After substitution of the expression given by eq. (3.15) into the functional (3.14), we find

$$S(\tilde{g}_1) = S_{WZNW}(\tilde{g}_1, k) - \frac{k^3 \sigma^2}{\pi} \int d^2 z \text{Tr} \left(\text{Tr } \hat{S} \tilde{g}_1^{-1} \partial \tilde{g}_1 \cdot \text{Tr } \hat{S} \tilde{g}_1^{-1} \bar{\partial} \tilde{g}_1 \right) + \mathcal{O}(\sigma^3). \quad (3.17)$$

In formula (3.17), we have used the equations of motion (2.5) in order to obtain the expression only in terms of the fields $\tilde{\phi}_1$ and \tilde{J}_1 , $\bar{\tilde{J}}_1$. Indeed, eq. (3.17) can be rewritten as follows

$$S(\tilde{g}_1) = S_{WZNW}(\tilde{g}_1, k) - \tau \int d^2 z \Sigma_{ab} \tilde{J}_1^a \bar{\tilde{J}}_1^b \tilde{\phi}_1^{b\bar{b}} + \mathcal{O}(\sigma^3), \quad (3.18)$$

where we have introduced the following notations

$$\tau = \frac{16k\sigma^2}{\pi}, \quad \Sigma_{ab} = \hat{S}^{a\bar{a}} \hat{S}^{b\bar{b}}. \quad (3.19)$$

The obtained representation for $S(\tilde{g}_1)$ in eq. (3.18) looks like an expansion in the coupling σ around the conformal theory described by the level k WZNW model $S(\tilde{g}_1, k)$. However, we cannot go on with the given perturbation of the unitary WZNW model, because the operator $L_{ab} \tilde{J}_1^a \bar{\tilde{J}}_1^b$ is irrelevant around this conformal model. In the next section, we will show that the perturbed theory in eq. (3.18) at special values of the Thirring coupling matrix $S_{a\bar{a}}$ can be understood as the infrared limit of a two-parameter perturbation around another conformal model in which the mentioned above operator becomes relevant.

4 Two-parametrical perturbation around the nonunitary WZNW model

Given a conformal level l WZNW model, we can build up the following composite operator [38]

$$O^{L,\bar{L}} = L_{ab}\bar{L}_{\bar{a}\bar{b}} : J^a \bar{J}^{\bar{a}} \phi^{b\bar{b}} :, \quad (4.20)$$

where

$$\begin{aligned} J &= J^a t^a = -\frac{l}{2} g^{-1} \partial g, \\ \bar{J} &= \bar{J}^{\bar{a}} t^{\bar{a}} = -\frac{l}{2} \bar{\partial} g g^{-1}, \\ \phi^{a\bar{a}} &= \text{Tr} : g^{-1} t^a g t^{\bar{a}} :. \end{aligned} \quad (4.21)$$

The product of the three operators in eq. (4.20) is defined according to [36]-[38]

$$O^{L,\bar{L}}(z, \bar{z}) = L_{ab}\bar{L}_{\bar{a}\bar{b}} \oint \frac{dw}{2\pi i} \oint \frac{d\bar{w}}{2\pi i} \frac{J^a(w) \bar{J}^{\bar{a}}(\bar{w}) \phi^{b\bar{b}}(z, \bar{z})}{|z - w|^2}, \quad (4.22)$$

where the product in the numerator of the integrand is understood as an OPE. It is easy to see that the given product does not contain singular terms provided the matrices L_{ab} and $\bar{L}_{\bar{a}\bar{b}}$ are symmetrical[‡].

By definition, the operator $O^{L,\bar{L}}$ is an affine descendant of ϕ . Indeed, $O^{L,\bar{L}}$ can be presented in the form

$$O^{L,\bar{L}}(0) = L_{ab}\bar{L}_{\bar{a}\bar{b}} J_{-1}^a \bar{J}_{-1}^{\bar{a}} \phi^{b\bar{b}}(0), \quad (4.23)$$

where

$$J_m^a = \oint \frac{dw}{2\pi i} w^m J^a(w), \quad \bar{J}_m^{\bar{a}} = \oint \frac{d\bar{w}}{2\pi i} \bar{w}^m \bar{J}^{\bar{a}}(\bar{w}). \quad (4.24)$$

Being an affine descendant, the operator $O^{L,\bar{L}}$ continues to be a Virasoro primary operator. Indeed, one can check that the state $O^{L,\bar{L}}(0)|0\rangle$ is a highest weight vector of the Virasoro

[‡]Indeed, the field ϕ is an affine primary vector. Therefore, its OPE with the affine current J is as follows

$$J^a(w) \phi^{b\bar{b}}(z, \bar{z}) = \frac{f^{abc}}{(w - z)} \phi^{c\bar{b}}(z, \bar{z}) + \text{regular terms}.$$

Substituting this formula in eq. (4.22), one can see that only regular terms will contribute provided L_{ab} is a symmetrical matrix.

algebra, with $|0\rangle$ the $SL(2, C)$ invariant vacuum. That is

$$L_0 O^{L, \bar{L}}(0)|0\rangle = \Delta_O O^{L, \bar{L}}(0)|0\rangle, \quad L_{m>0} O^{L, \bar{L}}(0)|0\rangle = 0. \quad (4.25)$$

Here the generators L_n are given by the contour integrals

$$L_n = \oint \frac{dw}{2\pi i} w^{n+1} T(w), \quad (4.26)$$

where $T(w)$ is holomorphic component of the affine-Sugawara stress-energy tensor of the conformal WZNW model,

$$T(z) = \frac{: J^a(z) J^a(z) :}{l + c_V}. \quad (4.27)$$

In eqs. (4.25), Δ_O is the conformal dimension of the operator $O^{L, \bar{L}}$. It is not difficult to find that

$$\Delta_O = \bar{\Delta}_O = 1 + \frac{c_V}{l + c_V}. \quad (4.28)$$

Here $\bar{\Delta}_O$ is the conformal dimension of $O^{L, \bar{L}}$ associated with antiholomorphic conformal transformations. The quantity c_V is defined according to

$$f^{acd} f^{bcd} = c_V \delta^{ab}. \quad (4.29)$$

From the formula for anomalous conformal dimensions of the operator $O^{L, \bar{L}}$, it is clear that when l is negative large (namely, when $-l > c_V$), the conformal dimensions are in the range between 0 and 1, $0 < \Delta_O = \bar{\Delta}_O < 1$. Hence, in this limit, the operator $O^{L, \bar{L}}$ becomes a relevant conformal operator. Correspondingly, for positive l the operator $O^{L, \bar{L}}$ is irrelevant.

In spite of the nonunitarity of the conformal WZNW model with negative level, the operator $O^{L, \bar{L}}$ corresponds to a unitary highest weight vector of the Virasoro algebra. Indeed, as we have shown above, $O^{L, \bar{L}}$ has positive conformal dimensions, whereas the Virasoro central charge of the nonunitary WZNW model in the large negative level limit is greater than one,

$$c_{WZNW}(l) = \frac{l \dim G}{l + c_V} = \dim G + \mathcal{O}(1/l) > 1. \quad (4.30)$$

Thus, the operator $O^{L, \bar{L}}$ lies in the unitary range of the Kac determinant and, hence, it provides a unitary representation of the Virasoro algebra.

Clearly, operators $O^{L,\bar{L}}$ with arbitrary symmetrical matrices L_{ab} , $\bar{L}_{\bar{a}\bar{b}}$ are Virasoro primary vectors with the same conformal dimensions. However, their fusion algebras may be different. Among all given operators, there are operators which obey the following fusion algebra [38]

$$O^{L,\bar{L}} \cdot O^{L,\bar{L}} = [O^{L,\bar{L}}] + [I] + \dots, \quad (4.31)$$

where the square brackets denote the contributions of $O^{L,\bar{L}}$ and identity operator I and their descendants, whereas dots stand for all other admitted operators with different conformal dimensions.

We have discovered [38] that the fusion given by eq. (4.32) results in the so-called master Virasoro equations [33],[34] for the matrices L , \bar{L} . For example, for the matrix L this equation is formulated as follows

$$L_{ab} = L_{ac}L_{cb} + \frac{2}{l} \left(L_{cd}L_{ef}f^{cea}f^{dfb} + L_{cd}f^{cdf}f^{dfa}L_{be} + L_{cd}f^{cef}f^{dfb}L_{ae} \right). \quad (4.32)$$

The matrix \bar{L} satisfies a similar equation. Note that the master equation has been originally obtained in the course of investigation of embeddings of the affine-Virasoro constructions into the affine algebra [33],[34]. Now we found that the same equation emerges in the fusion algebra of the *nonunitary* WZNW model. Due to the interesting flip of sign of the level in the fusion algebra, the unitary WZNW model gives rise to the master equation corresponding to a nonunitary affine-Virasoro construction [38].

In what follows, we will deal with the large $|l|$ limit. Therefore, it is interesting to look at the operator $O^{L,\bar{L}}$ in this limit. In the WZNW model, the given limit is the classical limit. This allows us to use classical expressions for the operators J , \bar{J} and ϕ . We find

$$O^{L,\bar{L}} \rightarrow G_{\mu\nu} \partial x^\mu \bar{\partial} x^\nu + B_{\mu\nu} \partial x^\mu \bar{\partial} x^\nu, \quad (4.33)$$

where x^μ are coordinates on the group manifold G , whereas

$$G_{\mu\nu} = -\frac{k^2}{8} L_{ab} \bar{L}_{\bar{a}\bar{b}} \phi^{b\bar{b}} e_{(\mu}^a \bar{e}_{\nu)}^{\bar{a}}, \quad B_{\mu\nu} = -\frac{k^2}{8} L_{ab} \bar{L}_{\bar{a}\bar{b}} \phi^{b\bar{b}} e_{[\mu}^a \bar{e}_{\nu]}^{\bar{a}}. \quad (4.34)$$

Here e_μ^a and $\bar{e}_\mu^{\bar{a}}$ define left- and right-invariant Killing vectors respectively. Note that when $L = \bar{L}$, the antisymmetric field $B_{\mu\nu} = 0$.

Thus, in the classical limit ($|l| \rightarrow \infty$), the operator $O^{L,\bar{L}}$ becomes the nonlinear sigma model. Therefore, the renormalizability of the sigma model together with the master

equation will guarantee the absence of dots on the right hand side of eq. (4.32) in the large $|l|$ limit.

Now we are going to demonstrate that given a solution of the master equation, we can construct four orthogonal to each other operators of the $O^{L,\bar{L}}$ type. It is known that if L_{ab} is a solution of the master equation, then the matrix

$$\tilde{L}_{ab} = \frac{l\delta^{ab}}{l - 2c_V} - L_{ab} \quad (4.35)$$

is also a solution of the same master equation [33]. Similarly, this will be true for \bar{L} . Let us show that the operator $O^{L,\bar{L}}$ is orthogonal to the operator $O^{\tilde{L},\bar{\tilde{L}}}$, where \tilde{L} is given by eq. (4.35). By computing the two point function, we find

$$\langle O^{L,\bar{L}}(1) O^{\tilde{L},\bar{\tilde{L}}}(0) \rangle = \frac{l^2}{4 \dim G} \left(L_{aa} - L_{ab} L_{ab} + \frac{2}{l} L_{ab} L_{cd} f^{ack} f^{bdk} \right) \bar{L}_{\bar{a}\bar{a}}. \quad (4.36)$$

But the expression in parenthesis vanishes due to the master equation. Thus, we arrive at

$$\langle O^{L,\bar{L}}(1) O^{\tilde{L},\bar{\tilde{L}}}(0) \rangle = 0. \quad (4.37)$$

In the same fashion, one can prove that there are two more operators, namely, $O^{L,\bar{\bar{L}}}$ and $O^{\tilde{\tilde{L}},\bar{\tilde{\tilde{L}}}}$, which are orthogonal both to each other and to the two operators just considered above. Obviously, all the four operators share the same dimensions and the similar fusion algebras. Therefore, all of them are appropriate for performing renormalizable perturbations around the nonunitary WZNW model. However, for our purposes, it will be sufficient to use only two operators $O^{L,\bar{L}}$ and $O^{\tilde{L},\bar{\tilde{L}}}$. It is necessary to point out that these operators are conjugated to each other under the following transformation

$$L \rightarrow \frac{lI}{l - 2c_V} - L. \quad (4.38)$$

Now we define a new theory

$$S(\epsilon, \tilde{\epsilon}) = S_{WZNW}(g, l) - \epsilon \int d^2z O(z, \bar{z}) - \tilde{\epsilon} \int d^2z \tilde{O}(z, \bar{z}), \quad (4.39)$$

where ϵ and $\tilde{\epsilon}$ are small parameters. From now on, we will omit superscripts on the perturbation operators.

We proceed to calculate the renormalization beta functions associated with the couplings ϵ , $\tilde{\epsilon}$. Away of criticality, where $\epsilon \neq 0$, $\tilde{\epsilon} \neq 0$, the renormalization group equations are given by [41]

$$\begin{aligned}\frac{d\epsilon}{dt} \equiv \beta &= (2 - 2\Delta_O)\epsilon - \pi C \epsilon^2 + \mathcal{O}^3(\epsilon, \tilde{\epsilon}), \\ \frac{d\tilde{\epsilon}}{dt} \equiv \tilde{\beta} &= (2 - 2\Delta_O)\tilde{\epsilon} - \pi \tilde{C} \tilde{\epsilon}^2 + \tilde{\mathcal{O}}^3(\epsilon, \tilde{\epsilon}),\end{aligned}\tag{4.40}$$

where the coefficients C , \tilde{C} are to be computed from the equations

$$\begin{aligned}\langle O(z_1, \bar{z}_1) O(z_2, \bar{z}_2) O(z_3, \bar{z}_3) \rangle &= C \|O\|^2 \Pi_{i < j}^3 \frac{1}{|z_{ij}|^{2\Delta_O}}, \\ \langle \tilde{O}(z_1, \bar{z}_1) \tilde{O}(z_2, \bar{z}_2) \tilde{O}(z_3, \bar{z}_3) \rangle &= \tilde{C} \|\tilde{O}\|^2 \Pi_{i < j}^3 \frac{1}{|z_{ij}|^{2\Delta_O}},\end{aligned}\tag{4.41}$$

with $z_{ij} = z_i - z_j$. Here

$$\|O\|^2 = \langle O(1) O(0) \rangle, \quad \|\tilde{O}\|^2 = \langle \tilde{O}(1) \tilde{O}(0) \rangle.\tag{4.42}$$

By using results of [37],[38], one can compute the coefficients C , \tilde{C} . We will be interested only in leading orders in $1/l$. To leading orders in $1/l$, the coefficients are given by

$$C = \frac{F^{abc} f^{abc} \bar{F}^{\bar{a}\bar{b}\bar{c}} f^{\bar{a}\bar{b}\bar{c}}}{c_V L_{dd}^{(0)} \bar{L}_{\bar{d}\bar{d}}^{(0)}} + \mathcal{O}(1/l), \quad \tilde{C} = \frac{\tilde{F}^{abc} f^{abc} \bar{\tilde{F}}^{\bar{a}\bar{b}\bar{c}} f^{\bar{a}\bar{b}\bar{c}}}{c_V \tilde{L}_{dd}^{(0)} \bar{\tilde{L}}_{\bar{d}\bar{d}}^{(0)}} + \tilde{\mathcal{O}}(1/l).\tag{4.43}$$

Here

$$\begin{aligned}F^{abc} &= f^{lmn} L_{al}^{(0)} L_{lm}^{(0)} L_{cn}^{(0)}, \\ \bar{F}^{\bar{a}\bar{b}\bar{c}} &= f^{\bar{l}\bar{m}\bar{n}} \bar{L}_{\bar{a}\bar{l}}^{(0)} \bar{L}_{\bar{l}\bar{m}}^{(0)} \bar{L}_{\bar{c}\bar{n}}^{(0)}, \\ \tilde{F}^{abc} &= f^{lmn} (\delta_{al} - L_{al}^{(0)}) (\delta_{bm} - L_{bm}^{(0)}) (\delta_{cn} - L_{cn}^{(0)}),\end{aligned}\tag{4.44}$$

where $L^{(0)}$, $\bar{L}^{(0)}$ are coefficients in the expansions of L , \bar{L} in $1/l$:

$$L_{ab} = L_{ab}^{(0)} + \mathcal{O}(1/l), \quad \bar{L}_{\bar{a}\bar{b}} = \bar{L}_{\bar{a}\bar{b}}^{(0)} + \bar{\mathcal{O}}(1/l^2).\tag{4.45}$$

The expressions for $L^{(0)}$, $\bar{L}^{(0)}$ can be obtained from the master equation. It has been found [42]

$$L_{ab}^{(0)} = \sum_c \Omega_{ac}^{(0)} \Omega_{bc}^{(0)} \theta^c.\tag{4.46}$$

There is a similar representation for $\bar{L}^{(0)}$. The quantities $\Omega^{(0)}$, θ are defined as follows

$$\Omega^{(0)}\Omega^{(0)T} = 1, \quad \theta^a = 0 \text{ or } 1, \quad a = 1, \dots, \dim G. \quad (4.47)$$

Besides, the quantities θ^a , $\Omega_{bc}^{(0)}$ are subject to the following quantization conditions [42]

$$0 = \sum_{cd} \theta^c (\theta^a + \theta^b - \theta^d) \hat{f}^{cda} \hat{f}^{cdb}, \quad a < b, \quad (4.48)$$

where

$$\hat{f}^{abc} = f^{lmn} \Omega_{al}^{(0)} \Omega_{bm}^{(0)} \Omega_{cn}^{(0)}. \quad (4.49)$$

Thus, given a solution of the master equation, we can derive the constants C and \tilde{C} and, correspondingly, fixed points ϵ^* , $\tilde{\epsilon}^*$ of the beta functions in eqs. (4.40). It is not difficult to find

$$\epsilon^* = -\frac{2c_V}{\pi C l}, \quad \tilde{\epsilon}^* = -\frac{2c_V}{\pi \tilde{C} l}. \quad (4.50)$$

By substitution of the obtained values of the coupling constants into the action in eq. (4.39), we come to the new conformal theory described by the following perturbative action

$$S(\epsilon^*, \tilde{\epsilon}^*) = S_{WZNW}(g, l) - \epsilon^* \int d^2 z O(z, \bar{z}) - \tilde{\epsilon}^* \int d^2 z \tilde{O}(z, \bar{z}). \quad (4.51)$$

Apparently, each pair of the matrices L , \bar{L} solving the master equation will lead to a CFT.

Let us consider the theory in eq. (4.51) with $\bar{L} = lI/(l - 2c_V)$. The last is the so-called Sugawara solution of the master equation. In this case the perturbation in eq. (4.51) mimics well the perturbation term in eq. (3.18). To make the resemblance between the two theories more transparent, we present the nonunitary WZNW model in the form [36]

$$S_{WZNW}(g, l) = S_{WZNW}(g, -l) - \frac{2}{\pi l} \int d^2 z : J^a \bar{J}^{\bar{a}} \phi^{a\bar{a}} :, \quad (4.52)$$

where $S_{WZNW}(g, -l)$ is the action of the unitary conformal level $-l$ WZNW model.

Then, the new perturbative conformal model can be rewritten as follows

$$S(\epsilon^*, \tilde{\epsilon}^*) = S_{WZNW}(g, -l) - \frac{2}{\pi(l - 2c_V)} \left[\left(1 - \frac{c_V}{\tilde{C}}\right) \delta^{ab} + \left(\frac{c_V}{\tilde{C}} - \frac{c_V}{C}\right) L_{ab}^{(0)} \right] \int d^2 z : J^a \bar{J}^{\bar{b}} \phi^{b\bar{b}} :. \quad (4.53)$$

Note that this is first order approximation to a certain exact conformal field theory.

Now we can compare the perturbative conformal model with the theory given by eq. (3.18). The two theories are renormalizable beyond conformal points. The first one described by eq. (3.18) is renormalizable because it originates from the renormalizable interaction of two WZNW models. The second theory is renormalizable by construction. Apparently, in leading orders in $1/l$ they coincide when $k = -l$ and

$$\tau\Sigma_{ab} = \frac{2c_V}{\pi(l - 2c_V)} \left[\left(\frac{1}{c_V} - \frac{1}{\bar{C}} \right) \delta^{ab} + \left(\frac{1}{\bar{C}} - \frac{1}{C} \right) L_{ab}^{(0)} \right]. \quad (4.54)$$

The last equation is, in fact, a condition of the conformal invariance of the theory described by formula (3.18). In its turn, this condition will fix conformal points of the system of two interacting WZNW models. The conformal values of the Thirring coupling matrix $S_{a\bar{a}}$ are defined from the following relation

$$S_{a\bar{a}}S_{b\bar{b}} = \frac{-c_V}{8k(k + 2c_V)} \left[\left(\frac{1}{c_V} - \frac{1}{\bar{C}} \right) \delta^{ab} + \left(\frac{1}{\bar{C}} - \frac{1}{C} \right) L_{ab}^{(0)} \right]. \quad (4.55)$$

In principle, it is possible to work out the higher order corrections in $1/l$ to the right hand side of eq. (4.55). But in this case, computations become much more cumbersome. Already the lowest approximation displays that there exist as many nontrivial conformal points in the theory of interacting WZNW models as the number of solutions to the master Virasoro equation. Note that the conformal theory at the conformal points given by eq. (4.55) is no longer based on the affine symmetry, since the perturbation operators are affine descendants but not affine primaries. Such a situation occurs also in conformal representations of the affine-Virasoro construction [43]. However, it is still very difficult to identify the perturbative conformal model described by the action in eq. (4.53) with one of many exact CFT's corresponding to the affine-Virasoro construction. We only mention that the obtained perturbative conformal model possesses the invariance under the so-called K-conjugation [33] which amounts to the transformation given by eq. (4.38). The classical action of the affine-Virasoro construction also enjoys this property [44].

All in all, we have proved that the theory of two interacting WZNW models with the coupling matrix $S_{a\bar{a}}$ obeying the condition (4.55) is the infrared limit of the perturbed nonunitary WZNW model described by eq. (4.51). In this connection, it is worthwhile

pointing out that among conformal solutions given by eq. (4.55) there are diagonal ones which resemble closely the conformal points of the generalized Thirring model found in [45].

While we have discussed the perturbation of the nonunitary WZNW model by two orthogonal operators $O^{L,\bar{L}}$ and $O^{\bar{L},\bar{L}}$, there are two more operators $O^{L,\bar{\bar{L}}}$ and $O^{\bar{L},\bar{\bar{L}}}$ which are also appropriate for performing renormalizable perturbations. However, the four-parameter perturbation around the nonunitary WZNW model, giving new CFT's, do not correspond to the system of two interacting WZNW models. Therefore, we do not consider perturbations of this type in the present paper.

5 Conclusion

Our main result is the equation (4.55) which gives the conditions of conformal invariance for the theory of two interacting WZNW models. This equation exhibits that there are as many conformal points as solutions to the master-Virasoro equation. Because it is very complicated, we could not identify the found perturbative conformal points with known exact CFT's. One substantial fact is that at the obtained conformal points the action of interacting WZNW models possesses the invariance under the K-conjugation. This permits us to believe that we deal actually with a certain approximation to the quantum affine-Virasoro action. However, the question still remains open.

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